

AN ELEMENTARY OBSTRUCTION TO A UNIFORM PRIME-GAP ASYMPTOTIC

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ABSTRACT. We formalize a uniform interpretation of the assertion that, if

$$d(x) = \max_{p_n < x} (p_{n+1} - p_n),$$

then for each fixed constant $C > 1$ one has

$$\pi(y + Cd(x)) - \pi(y) \sim \frac{Cd(x)}{\log y}$$

uniformly for $x/2 < y < x$. We show that this uniform assertion cannot hold simultaneously for two constants $C_1 < C_2$ with $0 < C_2 - C_1 < 1/2$. Consequently, the universal-in- C version of the assertion cannot be correct under this formalization.

1. INTRODUCTION

Let p_n denote the n -th prime, and write

$$g_n = p_{n+1} - p_n$$

for the corresponding prime gaps. For real $x > 2$, define

$$d(x) = \max_{p_n < x} (p_{n+1} - p_n).$$

Thus a prime gap is counted as soon as its left endpoint lies below x ; it is not required that the whole gap lie below x . We use the standard convention

$$\pi(t) = \#\{p \leq t : p \text{ prime}\}$$

for real t .

The motivating question is the following.

Question 1.1. Let $x/2 < y < x$ and $C > 1$. If

$$d = \max_{p_n < x} (p_{n+1} - p_n),$$

is it true that

$$\pi(y + Cd) - \pi(y) \sim \frac{Cd}{\log y} ?$$

This question appears as Bloom's Erdős Problem #1138 [1], with original source [2, 1.3]. It lies at the intersection of two classical themes: maximal prime gaps, going back to Cramér [3] and including the language of maximal gaps and first occurrences [4], and the distribution of primes in short intervals

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[5, 6, 7]. For modern large-gap context, see [8, 9]. We are not aware of a prior occurrence of the elementary obstruction below. It is distinct from Maier-type irregularity results for primes in short intervals; instead it exploits the fact that the definition

$$d(x) = \max_{p_n < x} (p_{n+1} - p_n)$$

counts a record prime gap whose right endpoint may lie beyond x .

We interpret the question uniformly in the variable y . For a fixed $C > 1$, let $A(C)$ denote the assertion

$$\sup_{x/2 < y < x} \left| \frac{\pi(y + Cd(x)) - \pi(y)}{Cd(x)/\log y} - 1 \right| \rightarrow 0 \quad (x \rightarrow \infty),$$

where y ranges over real numbers. The main point of this note is that this uniform interpretation is incompatible with requiring the assertion for every fixed constant $C > 1$.

Theorem 1.2. *Let*

$$1 < C_1 < C_2, \quad 0 < C_2 - C_1 < \frac{1}{2}.$$

Then $A(C_1)$ and $A(C_2)$ cannot both hold.

2. PROOF OF THE THEOREM

Proof of Theorem 1.2. Set

$$\eta = C_2 - C_1.$$

Thus $0 < \eta < 1/2$. Prime gaps are unbounded: for each M , the integers

$$(M+1)! + 2, (M+1)! + 3, \dots, (M+1)! + (M+1)$$

are all composite. Therefore the sequence of running maxima of the gaps increases infinitely often. Hence there are infinitely many strict record gaps. Choose indices n_k such that

$$g_{n_k} > g_m \quad (m < n_k).$$

Write

$$P_k = p_{n_k}, \quad Q_k = p_{n_k+1}, \quad D_k = Q_k - P_k = g_{n_k}.$$

Then D_k is larger than every earlier prime gap.

Choose

$$\alpha \in \left(\frac{1}{2} + \eta, 1 \right),$$

which is possible because $\eta < 1/2$, and define

$$x_k = P_k + \alpha D_k, \quad y_k = P_k + \frac{1}{2} D_k, \quad z_k = P_k + \left(\frac{1}{2} + \eta \right) D_k.$$

Since $1/2 < 1/2 + \eta < \alpha < 1$, we have

$$P_k < y_k < z_k < x_k < Q_k.$$

Moreover,

$$2y_k - x_k = 2 \left(P_k + \frac{1}{2} D_k \right) - (P_k + \alpha D_k) = P_k + (1 - \alpha) D_k > 0.$$

Thus

$$\frac{x_k}{2} < y_k < z_k < x_k.$$

So both y_k and z_k are admissible choices in the range appearing in $A(C)$ with $x = x_k$.

Next we compute $d(x_k)$. Since

$$P_k < x_k < Q_k$$

and there are no primes in (P_k, Q_k) , we have

$$p_n < x_k \iff n \leq n_k.$$

Therefore

$$d(x_k) = \max_{p_n < x_k} (p_{n+1} - p_n) = \max_{1 \leq n \leq n_k} g_n.$$

The maximum includes the record gap

$$g_{n_k} = D_k,$$

and, by the strict record property, all earlier gaps are smaller. Hence

$$d(x_k) = D_k.$$

Because y_k and z_k both lie strictly inside the prime-free interval (P_k, Q_k) , there are no primes in $(y_k, z_k]$. Hence

$$\pi(y_k) = \pi(z_k) = \pi(P_k).$$

On the other hand,

$$y_k + C_2 D_k = P_k + \left(\frac{1}{2} + C_2 \right) D_k,$$

whereas

$$z_k + C_1 D_k = P_k + \left(\frac{1}{2} + \eta + C_1 \right) D_k = P_k + \left(\frac{1}{2} + C_2 \right) D_k.$$

Thus the two right endpoints are identical. If we put

$$R_k = y_k + C_2 D_k = z_k + C_1 D_k,$$

then

$$\begin{aligned} \pi(y_k + C_2 D_k) - \pi(y_k) &= \pi(R_k) - \pi(y_k) \\ &= \pi(R_k) - \pi(z_k) \\ &= \pi(z_k + C_1 D_k) - \pi(z_k). \end{aligned}$$

Call this common quantity N_k .

Since $n_k \rightarrow \infty$, we have $P_k \rightarrow \infty$, and therefore $x_k \rightarrow \infty$. If $A(C_2)$ holds, then the admissibility of y_k and $d(x_k) = D_k$ give

$$\frac{N_k}{C_2 D_k / \log y_k} \longrightarrow 1.$$

If $A(C_1)$ holds, then the admissibility of z_k and $d(x_k) = D_k$ give

$$\frac{N_k}{C_1 D_k / \log z_k} \longrightarrow 1.$$

Consequently the two main terms must be asymptotic to one another:

$$\frac{C_2 D_k / \log y_k}{C_1 D_k / \log z_k} \longrightarrow 1.$$

Equivalently,

$$\frac{C_2}{C_1} \cdot \frac{\log z_k}{\log y_k} \longrightarrow 1.$$

It remains only to compare the logarithms. We have

$$\frac{z_k}{y_k} = \frac{P_k + \left(\frac{1}{2} + \eta\right) D_k}{P_k + \frac{1}{2} D_k} = 1 + \frac{\eta D_k}{P_k + \frac{1}{2} D_k}.$$

Since $P_k > 0$,

$$1 < \frac{z_k}{y_k} < 1 + 2\eta < 2.$$

Therefore

$$\log z_k = \log y_k + \log \left(\frac{z_k}{y_k} \right) = \log y_k + O(1).$$

Since $P_k \rightarrow \infty$, also $y_k \rightarrow \infty$, and so

$$\frac{\log z_k}{\log y_k} \longrightarrow 1.$$

Hence

$$\frac{C_2}{C_1} \cdot \frac{\log z_k}{\log y_k} \longrightarrow \frac{C_2}{C_1} \neq 1,$$

contradicting the conclusion obtained from the two assumed asymptotics. Thus $A(C_1)$ and $A(C_2)$ cannot both hold. \square

3. CONSEQUENCE

Corollary 3.1. *The assertion $A(C)$ cannot hold for every fixed $C > 1$.*

Proof. If $A(C)$ held for every fixed $C > 1$, then it would hold in particular for any pair $1 < C_1 < C_2$ with $0 < C_2 - C_1 < 1/2$, contradicting Theorem 1.2. \square

Remark 3.2. We interpret the question as asking whether the displayed asymptotic holds uniformly for all real y with $x/2 < y < x$, for every fixed constant $C > 1$. Under this natural universal-in- C reading, the answer is negative: Theorem 1.2 shows that the assertion cannot hold simultaneously for two constants $1 < C_1 < C_2$ with $C_2 - C_1 < 1/2$, and hence cannot hold for all $C > 1$.

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